

REDUCTION OF CONTROLLED LAGRANGIAN AND HAMILTONIAN SYSTEMS WITH SYMMETRY*

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Abstract. We develop reduction theory for controlled Lagrangian and controlled Hamiltonian systems with symmetry. Reduction theory for these systems is needed in a variety of examples, such as a spacecraft with rotors, a heavy top with rotors, and underwater vehicle dynamics. One of our main results shows the equivalence of the method of reduced controlled Lagrangian systems and that of reduced controlled Hamiltonian systems in the case of simple mechanical systems with symmetry.

Key words. controlled Lagrangian systems, controlled Hamiltonian systems, reduction, equivalence

AMS subject classifications. 70Q05, 93C15, 34H05

DOI. 10.1137/S0363012902412951

1. Introduction. We develop the theory of symmetry reduction for the class of controlled Lagrangian (CL) and controlled Hamiltonian (CH) systems with symmetry and show the equivalence of the method of reduced CL systems and that of reduced CH systems for simple mechanical systems. The phrases “controlled Lagrangian” and “controlled Hamiltonian” systems were coined in [20], and their technical definition is recalled below. The concept of a CH system used here is the same as that of a “port-controlled Hamiltonian system” in [32]. We prefer to use the term “controlled Hamiltonian” in a way that is parallel to “controlled Lagrangian.” The notion of an “implicit Hamiltonian system” is related to that of a CH system but is based on the more general notion of Dirac structures rather than symplectic and Poisson structures, and a type of reduction theory (different from what is given here) in that context was developed in [6]. The notion of an “implicit Lagrangian system” is developed, also using Dirac structures, in [35], and in future work we hope to explore the reduction theory of controlled implicit Hamiltonian and Lagrangian systems and the relation between them.

CL systems. We will begin by describing the Lagrangian side. To help design stabilizing controllers for mechanical systems whose equations are written in Euler–Lagrange form with control forces, the CL method was developed by a number of authors, including [3], [4], [7], [10], [11], [12], [13], [14], [15], [19], [20], [22], [23], [33]. In the study of CL systems, it is important to take into account external forces and control forces, including gyroscopic forces.

The main idea of the CL method is to proceed in the following steps:

1. Define a (feedback-transformation) equivalence relation among CL systems.
2. Given a CL system, find an equivalent CL system such that the energy of the second CL system has a minimum at the equilibrium of interest and the system is forced or controlled by a dissipative (plus gyroscopic) force.

*Received by the editors August 10, 2002; accepted for publication (in revised form) October 23, 2003; published electronically June 15, 2004. This research was partially supported by Caltech and by ONR contract N00014-02-1-0826.

<http://www.siam.org/journals/sicon/43-1/41295.html>

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3. Using the energy as a Lyapunov function, obtain asymptotic stability of the equilibrium in the second CL system.
4. Since the two systems are equivalent, the equivalence relation gives an asymptotically stabilizing controller for the first system.

CH systems. A similar approach has been taken on the Hamiltonian side, too, where the main governing equations are in Hamiltonian form (see [5], [9], [32], [34] and references therein).

Equivalence. There have been some recent studies concerning the equivalence of both methods. First of all, [5] showed that the CH method is more general than the CL method for simple mechanical systems and used a limited definition of CL systems. Then [20] extended the theory of CL systems and showed the full equivalence of the method of CL systems with the method of CH systems for simple mechanical systems.

Symmetry. The main goal of the present paper is to include symmetry in the discussion. We do so in a systematic and general way. In previous works, symmetry was considered in certain cases (see, for instance, [15], [23], and [26]), but a comprehensive theory has been lacking.

In this paper, we define the notion of G -invariant CL and CH systems and develop a reduction theory for them. In addition, we trace through the equivalence of the CL and CH systems under the reduction process.

Reduction theory. Reduction theory for mechanical systems has been studied both on the Lagrangian side and on the Hamiltonian side. It has proven to be a powerful tool for studying the dynamics, stability, and control of many mechanical systems, such as rigid body and spacecraft systems, underwater vehicles, heavy tops, and fluid systems.

On the Lagrangian side, one reduces variational principles, an idea due to [29] and [30]. A comprehensive account of this theory is given in [17]. On the other hand, on the Hamiltonian side, one reduces symplectic and Poisson structures. See, for instance, [1], [27], and [28] and references therein. We shall merge these ideas with those of CL and CH systems to develop the notions and basic properties of reduced CL and CH systems.

Equivalence and reduction. Once we have developed the reduction theory, we will be in a position to define an equivalence relation among G -invariant CL (resp., CH) systems; in this context, we show that two reduced CL (resp., CH) systems are equivalent if and only if their unreduced G -invariant CL (resp., CH) systems are equivalent. This is given in Theorems 2.10 and 3.9.

We also show that this reduced equivalence relation is a feedback-transform equivalence relation (see Theorems 2.11 and 3.10). Based on these results and the equivalence of the methods of CL and CH systems for simple mechanical systems proven in [20], we prove that the method of reduced CL systems and that of reduced CH systems are equivalent for reduced simple mechanical systems (this is given in Theorem 4.3).

Example. In the final section, we review the example of a satellite controlled via a spinning rotor and show that the Euler–Poincaré matching conditions derived in [8] and [15] directly in the reduced context can be obtained in a natural way through the reduction process. We study the stabilization of the heavy top with rotors, using the reduced CL method as an illustration.

2. Reduction of CL systems with symmetry. We begin this section with some general notation, then we introduce the notion of a CL system with symmetry, and finally study the theory of reduction for such systems.

We first summarize some general notation that will be used. We refer to [2], [17], [25], and [28] for more details and background information.

Manifolds and bundles. Let Q be the configuration manifold, and let $\tau_Q : TQ \rightarrow Q$ and $\pi_Q : T^*Q \rightarrow Q$ be the tangent bundle projection and the cotangent bundle projection, respectively. The second order tangent bundle $\tau_Q^{(2)} : T^{(2)}Q \rightarrow Q$ is defined as follows. For $\bar{q} \in Q$, elements of $T_q^{(2)}Q$ are equivalence classes of curves in Q , the equivalence relation being defined as follows: two curves $q_i(t)$, $i = 1, 2$, with $q_1(\bar{t}) = q_2(\bar{t}) = \bar{q}$, are equivalent, by definition, if and only if in any local chart we have $q_1^{(l)}(\bar{t}) = q_2^{(l)}(\bar{t})$ for $l = 1, 2$, where $q^{(l)}(t)$ denotes the derivative of order l . The second order bundle has local coordinates given by $(q^i, \dot{q}^i, \ddot{q}^i)$ and may be thought of as a subbundle of the second tangent bundle TTQ via the embedding $(q^i, \dot{q}^i, \ddot{q}^i) \mapsto (q^i, \dot{q}^i, \ddot{q}^i, \ddot{q}^i)$. As is explained in [28], the second order tangent bundle is the basic space on which the Euler–Lagrange operator of mechanics is defined; in fact, given a Lagrangian L , the associated Euler–Lagrange operator \mathcal{EL} induces a bundle map $\mathcal{EL}(L) : T^{(2)}Q \rightarrow T^*Q$.

For a manifold M , $\mathcal{F}(M)$ denotes the set of smooth real-valued functions on M .

Symmetry groups. Let G be a Lie group acting (on the left) on Q freely and properly so that $\pi_G(Q) : Q \rightarrow Q/G$ becomes a principal bundle. The tangent (resp., cotangent) lift of the action of G on Q defines an action of G on TQ (resp., T^*Q), which is automatically free and proper as well, so that the maps $\tau_{/G} : TQ \rightarrow TQ/G$ (resp., $\pi_{/G} : T^*Q \rightarrow T^*Q/G$) also define principal bundles. When M is a manifold on which G acts, we let $[m]_G$ denote an equivalence class of $m \in M$ in the quotient space M/G . Even though we do not explicitly denote the manifold M in this notation, it will be clear which manifold is meant from the context. The space TQ/G becomes a vector bundle with base Q/G by inheriting the vector bundle structure of TQ as follows:

$$[u_q]_G + \lambda[v_q]_G = [u_q + \lambda v_q]_G,$$

where $\lambda \in \mathbb{R}$; $u_q, v_q \in T_qQ$; and $[u_q]_G, [v_q]_G$ are their equivalence classes in the quotient space TQ/G . The fiber $(TQ/G)_x$ is isomorphic, as a vector space, to T_qQ for each $x = [q]_G \in Q/G$, $q \in Q$ (see Lemma 2.4.1 in [17]). In the same manner, the space T^*Q/G becomes a vector bundle with base Q/G .

Vertical lifts. Let V be a vector bundle over a manifold Q . The vertical lift of a vector $w_q \in V_q$ along the vector $v_q \in V_q$ is the vector, $\text{vlift}_{v_q}(w_q) \in T_{v_q}V$, defined by

$$\text{vlift}_{v_q}(w_q) = \left. \frac{d}{dt} \right|_{t=0} (v_q + tw_q).$$

The vertical lift of a fiber-preserving map $F : V \rightarrow V$ is a section, $\text{vlift}(F) : V \rightarrow TV$, defined by

$$(2.1) \quad \text{vlift}(F)(v_q) = \text{vlift}_{v_q}(F(v_q)).$$

The vertical lift of a subbundle W of V is the subbundle of TV defined by

$$(2.2) \quad \text{vlift}(W) = \{\text{vlift}_{v_q}(w_q) \mid v_q \in V_q, w_q \in W_q, q \in Q\}.$$

CL systems. We next recall the definition of a CL system from [20].

DEFINITION 2.1. A controlled Lagrangian (CL) system is a triple (L, F, W) , where the function $L : TQ \rightarrow \mathbb{R}$ is the Lagrangian, the fiber-preserving map $F :$

$TQ \rightarrow T^*Q$ is the (external) force map, and the subbundle W of T^*Q is called the control subbundle. When one chooses a feedback control law $u : TQ \rightarrow W$, the triple (L, F, u) will denote the closed-loop CL system.

Euler–Lagrange operator. The Euler–Lagrange operator \mathcal{EL} assigns to a Lagrangian $L : TQ \rightarrow \mathbb{R}$ a bundle map $\mathcal{EL}(L) : T^{(2)}Q \rightarrow T^*Q$, which may be written in local coordinates (and using the index summation convention) as

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) - \frac{\partial L}{\partial q^i}(q, \dot{q}) \right) \mathbf{d}q^i,$$

in which it is understood that one regards the first term on the right-hand side as a function on the second order tangent bundle $T^{(2)}Q$ by formally applying the chain rule and replacing dq/dt by \dot{q} everywhere. The equations of motion of a CL system (L, F, W) with a choice of feedback control $u : TQ \rightarrow W$ are written as

$$(2.3) \quad \mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = F(q, \dot{q}) + u(q, \dot{q}),$$

which may be derived from the Lagrange–d’Alembert principle.

CL systems with symmetry. We next define the notion of a G -invariant CL system on TQ and the associated notion of a reduced CL system on TQ/G , where G is a Lie group acting on Q .

DEFINITION 2.2. *Let G be a Lie group action on Q . A G -invariant controlled Lagrangian (G -CL) system is a CL system, (L, F, W) , where L is a G -invariant Lagrangian, F is a G -equivariant force map, and W is a G -invariant subbundle of T^*Q .*

Reduction of CL systems with symmetry. Now we can define the notion of a reduced CL system.

DEFINITION 2.3. *A reduced controlled Lagrangian (RCL) system is a triple (l, f, U) , where the function $l : TQ/G \rightarrow \mathbb{R}$ is called the reduced Lagrangian, the fiber-preserving map $f : TQ/G \rightarrow T^*Q/G$ is called the reduced force map, and the subbundle U of T^*Q/G is called the reduced control subbundle. A feedback control for an RCL system is a fiber-preserving map of TQ/G into U .*

Suppose that we are given a G -CL system (L, F, W) . The G -invariance of L induces the reduced Lagrangian l on TQ/G satisfying

$$(2.4) \quad l \circ \tau_{/G} = L.$$

The equivariance of F induces the reduced force map $[F]_G : TQ/G \rightarrow T^*Q/G$ satisfying

$$(2.5) \quad [F]_G \circ \tau_{/G} = \pi_{/G} \circ F.$$

Similarly, a G -invariant control subbundle W induces a reduced control subbundle W/G in a natural way; namely, we have $W = \pi_G^{-1}(W/G)$.

These considerations lead to the following definition.

DEFINITION 2.4. *The RCL system of a G -CL system (L, F, W) is the triple $(l, [F]_G, W/G)$, where l is the reduced Lagrangian satisfying (2.4) and $[F]_G$ is the reduced force satisfying (2.5).*

One may ask whether there exists a G -CL system on TQ when one is given a RCL system on TQ/G . The following proposition proves its existence and uniqueness.

PROPOSITION 2.5. *Given a RCL system (l, f, U) , there is a unique G -CL system (L, F, W) whose RCL system is (l, f, U) .*

Proof. Define L by (2.4). Define a force map F on TQ as follows: for $v_q, w_q \in T_q Q$,

$$(2.6) \quad \langle F(v_q), w_q \rangle = \langle f \circ \tau_{/G}(v_q), \tau_{/G}(w_q) \rangle.$$

One can check the G -equivariance of F . One can also check that relation (2.6) defines the unique *fiber-preserving* map F of TQ to T^*Q . Let $W := \pi_{/G}^{-1}(U)$. By construction, (L, F, W) is the unique G -CL system whose RCL system is (l, f, U) . \square

By Proposition 2.5, we can write an arbitrary RCL in the form of the RCL of a G -CL without loss of generality. Additionally, the proof of Proposition 2.5 establishes the following assertion: given a fiber-preserving map $f : TQ/G \rightarrow T^*Q/G$, there exists the unique fiber-preserving map $F : TQ \rightarrow T^*Q$ satisfying

$$f \circ \tau_{/G} = \pi_{/G} \circ F.$$

Reduced equations of motion. Given a G -CL system (L, F, W) , the G -invariance of L implies the G -equivariance of the bundle map $\mathcal{EL}(L) : T^{(2)}Q \rightarrow T^*Q$, which induces a quotient map

$$\mathcal{REL}(l) := [\mathcal{EL}(L)]_G : T^{(2)}Q/G \rightarrow T^*Q/G,$$

which depends only on the reduced Lagrangian l on TQ/G induced from L . The operator \mathcal{REL} is called the reduced Euler–Lagrange operator. The equations of motion of an RCL $(l, [F]_G, W/G)$ with a choice of control $[u]_G : TQ/G \rightarrow W/G$ are given by

$$\mathcal{REL}(l)([q, \dot{q}, \ddot{q}]_G) = [F]_G([q, \dot{q}]_G) + [u]_G([q, \dot{q}]_G).$$

To write computable equations of \mathcal{REL} , one normally chooses a principal connection on the principal bundle $Q \rightarrow Q/G$ to identify the quotient bundles,

$$\begin{aligned} TQ/G &\text{ with } T(Q/G) \oplus \tilde{\mathfrak{g}}, \\ T^{(2)}Q/G &\text{ with } T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}}, \end{aligned}$$

and

$$T^*Q/G \text{ with } T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

where $\tilde{\mathfrak{g}} := Q \times_G \mathfrak{g}$ is the associated adjoint bundle, $\tilde{\mathfrak{g}}^* := Q \times_G \mathfrak{g}^*$ is the associated coadjoint bundle, $T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}}$ is the product bundle over Q/G , $2\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$, and \oplus is the Whitney sum (see Lemmas 2.4.2 and 3.2.2 in [17]). For example, a principal connection $A : TQ \rightarrow \mathfrak{g}$ on the principal bundle $\pi : Q \rightarrow Q/G$ induces the bundle isomorphism $\alpha_A : TQ/G \rightarrow T(Q/G) \oplus \tilde{\mathfrak{g}}$ as follows:

$$(2.7) \quad \alpha_A([q, \dot{q}]_G) = T\pi(q, \dot{q}) \oplus [q, A(q, \dot{q})]_G.$$

With these identifications, \mathcal{REL} induces the Lagrange–Poincaré operator \mathcal{LP} as follows: for a reduced Lagrangian l ,

$$(2.8) \quad \mathcal{LP}(l) : T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}} \rightarrow T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

Hence, the reduced Euler–Lagrange operator \mathcal{REL} may be replaced by the Lagrange–Poincaré operator \mathcal{LP} in this paper as far as one chooses a connection on $Q \rightarrow Q/G$. Further details may be found in [17].

Let us give the local coordinate expression of the Lagrange–Poincaré operator \mathcal{LP} induced from a connection A on the principal bundle $Q \rightarrow Q/G$. This will be used in section 5. We choose a local trivialization of the bundle $Q \rightarrow Q/G$ to be $X \times G \rightarrow X$, where X is an open subset of \mathbb{R}^r with $r = \dim(Q/G)$. Then, at any tangent vector $(x, g, \dot{x}, \dot{g}) \in T_{(x,g)}(X \times G)$, we have

$$A(x, g, \dot{x}, \dot{g}) = \text{Ad}_g(A_e(x) \cdot \dot{x} + \xi),$$

where A_e is the \mathfrak{g} -valued 1-form on X defined by $A_e(x) \cdot \dot{x} = A(x, e, \dot{x}, 0)$ and $\xi = g^{-1}\dot{g}$. The bundle isomorphism α_A in this case becomes

$$\alpha_A([x, g, \dot{x}, \dot{g}]_G) = (x, \dot{x}) \oplus (x, \Omega),$$

where $\Omega = A_e(x) \cdot \dot{x} + \xi$ and we choose a local trivialization of the associated bundle $\tilde{\mathfrak{g}}$ to be $X \times \mathfrak{g} \rightarrow X$. The Lagrange–Poincaré operator \mathcal{LP} for a reduced Lagrangian l on $TQ/G = T(Q/G) \oplus \tilde{\mathfrak{g}}$ gives

$$(2.9) \quad \mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt} \frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} + \frac{\partial l}{\partial \Omega^a} (B_{\beta\alpha}^a \dot{x}^\beta + C_{ab}^a \Omega^d A_\alpha^b) \\ \frac{d}{dt} \frac{\partial l}{\partial \Omega^b} - \frac{\partial l}{\partial \Omega^a} (C_{ab}^a \Omega^d - C_{db}^a A_\alpha^d \dot{x}^\alpha) \end{pmatrix},$$

where $B_{\beta\alpha}^a$ is the curvature of the connection $A_e = (A_\alpha^a)$, C_{bd}^a are the structure constants of the Lie algebra \mathfrak{g} , and $\Omega = (\Omega^a)$ and $x = (x^\alpha)$ with $a = 1, \dots, \dim \mathfrak{g}$ and $\alpha = 1, \dots, \dim(Q/G)$. More details of the derivation of (2.9) may be found in sections 3.3 and 4.2 of [17]. In particular, if one chooses a local trivial connection, i.e., $A_e = 0$, then the Lagrange–Poincaré equation in (2.9) is given by

$$(2.10) \quad \mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt} \frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} \\ \frac{d}{dt} \frac{\partial l}{\partial \xi^b} - C_{ab}^a \xi^d \frac{\partial l}{\partial \xi^a} \end{pmatrix}.$$

We briefly mention the relation between trajectories of G -CL systems and trajectories of RCL systems. Let (L, F, W) be a G -CL system and $(l, [F]_G, W/G)$ its RCL system. Choose an arbitrary G -equivariant feedback control law $u : TQ \rightarrow W$ for (L, F, W) . The control u induces a reduced map $[u]_G : TQ/G \rightarrow W/G$. Then, if $(q(t), \dot{q}(t)) \in TQ$ is a trajectory of the closed-loop system (L, F, u) , then $\tau_{/G}(q(t), \dot{q}(t)) \in TQ/G$ is the trajectory of the closed-loop system $(l, [F]_G, [u]_G)$.

Simple CL systems. We define simple CL systems, which include most mechanical systems in engineering applications.

DEFINITION 2.6. A CL system (L, F, W) on TQ is called simple if its Lagrangian $L : TQ \rightarrow \mathbb{R}$ is of the form kinetic minus potential energy as follows:

$$(2.11) \quad L(q, \dot{q}) = \frac{1}{2} m_q(\dot{q}, \dot{q}) - V(q),$$

where m is a (generalized) mass tensor, i.e., a nondegenerate symmetric $(0, 2)$ -tensor. A reduced CL system $(l, [F]_G, W/G)$ is called simple if the reduced Lagrangian l is induced by a G -invariant simple Lagrangian L on TQ . The acronym (R)SCL will denote “(reduced) simple controlled Lagrangian.”

When a simple G -invariant Lagrangian L is given by (2.11), its RSCL $l : TQ/G \rightarrow \mathbb{R}$ is given by

$$l([q, \dot{q}]_G) = \frac{1}{2}[m]_G([q, \dot{q}]_G, [q, \dot{q}]_G) - [V]_G([q]_G),$$

where $[m]_G \in \Gamma(Q/G, T^*Q/G \otimes T^*Q/G)$ is the reduced mass tensor induced from the G -invariance of the mass tensor $m \in \Gamma(Q, T^*Q \otimes T^*Q)$ and $[V]_G : Q/G \rightarrow \mathbb{R}$ is the reduced potential energy.

SCL-equivalence. We now recall the fundamental definition of CL-equivalence from [20] as follows.

DEFINITION 2.7. *Two SCL systems (L_1, F_1, W_1) and (L_2, F_2, W_2) are said to be CL-equivalent, or simply, $(L_1, F_1, W_1) \stackrel{L}{\sim} (L_2, F_2, W_2)$, if the following Euler-Lagrange matching conditions hold:*

$$\text{ELM-1: } W_1 = m_1 m_2^{-1}(W_2),$$

$$\text{ELM-2: } \text{Im}[\mathcal{EL}(L_1) - F_1 - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)] \subset W_1,$$

where m_i is the mass tensor of L_i and Im means the pointwise image of the map in brackets.

The following proposition from [20] explains the significance of the CL-equivalence property. It shows that in a very natural sense the two control systems can be made to correspond by using an appropriate choice of control.

PROPOSITION 2.8. *Suppose that two SCL systems (L_i, F_i, W_i) , $i = 1, 2$, are CL-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two control laws u_i , $i = 1, 2$, is given by*

$$(2.12) \quad u_1 = \mathcal{EL}(L_1) - F_1 - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2) + m_1 m_2^{-1} u_2,$$

where m_i is the mass tensor of L_i , $i = 1, 2$.

Proof. Recall that the Euler-Lagrange operator is given by

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q})_j = m_{ij} \ddot{q}^i + \frac{\partial m_{ij}}{\partial q^k} \dot{q}^i \dot{q}^k - \frac{1}{2} \frac{\partial m_{ik}}{\partial q^j} \dot{q}^i \dot{q}^k + \frac{\partial V}{\partial q^j},$$

where the Lagrangian L is given by

$$L = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j - V(q).$$

One can solve (2.3) for \ddot{q} . Denote by \ddot{q}_{L_i} the expression of the acceleration \ddot{q} obtained from the closed-loop SCL system (L_i, F_i, u_i) , $i = 1, 2$. Then,

$$m_1(\ddot{q}_{L_1} - \ddot{q}_{L_2}) = u_1 - m_1 m_2^{-1} u_2 - [(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)].$$

The conditions ELM-1 and ELM-2 imply that (2.12) holds if and only if $\ddot{q}_{L_1} = \ddot{q}_{L_2}$ if and only if they produce the same equations of motion. Notice that the term

$$\mathcal{EL}(L_1) - m_1 m_2^{-1} \mathcal{EL}(L_2)$$

in (2.12) can be regarded as a map defined on TQ because the acceleration \ddot{q} cancels out. \square

RSCL-equivalence. We now define an equivalence relation among RSCL systems on TQ/G .

DEFINITION 2.9. *Two RSCL systems $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$, are said to be reduced-CL-equivalent (RCL-equivalent), or simply*

$$(l_1, [F_1]_G, W_1/G) \stackrel{L}{\sim} (l_2, [F_2]_G, W_2/G),$$

if the following reduced Euler-Lagrange matching conditions hold:

$$\text{RELM-1: } W_1/G = [m_1]_G [m_2]_G^{-1} (W_2/G),$$

$$\text{RELM-2: } \text{Im} [\mathcal{REL}(l_1) - [F_1]_G - [m_1]_G [m_2]_G^{-1} (\mathcal{REL}(l_2) - [F_2]_G)] \subset W_1/G,$$

where $[m_i]_G$ is the reduced mass tensor of l_i , $i = 1, 2$, and Im means the pointwise image of the map in brackets.

Equivalence commutes with reduction. The following theorem explains the relationship between the CL-equivalence relation among G -SCL systems and the RCL-equivalence relation among RSCL systems.

THEOREM 2.10. *Two G -SCL systems are CL-equivalent if and only if their associated RSCL systems are RCL-equivalent.*

Proof. Let (L, F, W) be a G -SCL system, and $(l, [F]_G, W/G)$ its associated RSCL system. Then, the theorem follows from the G -invariance of W and the following relations:

$$\mathcal{REL}(l) \circ \tau_{/G}^{(2)} = \pi_{/G} \circ \mathcal{EL}(L), \quad [F]_G \circ \tau_{/G} = \pi_{/G} \circ F,$$

where $\tau_{/G}^{(2)} : T^{(2)}Q \rightarrow T^{(2)}Q/G$ is the G -quotient map. \square

Hence, one can check the RCL-equivalence of two RSCL systems in two ways; one is to directly check it, and the other is to check the CL-equivalence of their associated *unreduced* G -SCL systems. In practice, it is more convenient to check it directly at the reduced level; we shall see an example of this in section 5.

The following theorem explains the property of the RCL-equivalence relation.

THEOREM 2.11. *Suppose that two RSCL systems $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$, are RCL-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two control laws $[u_i]_G$, $i = 1, 2$, is given by*

$$(2.13) \quad \begin{aligned} [u_1]_G &= \mathcal{REL}(l_1) - [F_1]_G - [m_1]_G [m_2]_G^{-1} (\mathcal{REL}(l_2) - [F_2]_G) \\ &\quad + [m_1]_G [m_2]_G^{-1} [u_2]_G, \end{aligned}$$

where $[m_i]_G$ is the reduced mass tensor of l_i , $i = 1, 2$.

Proof. Let $[u_i]_G$ be a control for $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$. Let (L_i, F_i, W_i) be the unreduced G -CL system of $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$. By Theorem 2.10, the two G -SCL systems are CL-equivalent. By Proposition 2.8, the two closed-loop G -SCL systems (L_i, F_i, u_i) , $i = 1, 2$, produce the same equations of motion when u_1 and u_2 satisfy (2.12). Hence, the two closed-loop RSCL systems $(l_i, [F_i]_G, [u_i]_G)$, $i = 1, 2$, produce the same equations of motion when $[u_1]_G$ and $[u_2]_G$ satisfy (2.13), because each term in (2.12) is G -equivariant. Also notice that, for any choice of $[u_i]_G$, one can choose the other $[u_j]_G$ such that (2.13) holds. \square

One can prove Theorem 2.11 by comparing the expressions of “accelerations” of both equations using (2.9), as in the proof of Proposition 2.8. For this purpose, one needs to choose a connection on $Q \rightarrow Q/G$ because one has to split the variations

to write down the equations of motion in coordinates, because the Euler–Lagrange equations come from the variational principles (see (2.9) of this paper and Chapter 3 of [17] for more detail). In the current proof of Theorem 2.11, we were able to bypass this route by Theorem 2.10.

3. Reduction of CH systems with symmetry. There is a Hamiltonian counterpart to CL systems called *CH systems*. We study the reduction of CH systems with symmetry in this section.

CH systems. We first recall the definition of CH systems from [20].

DEFINITION 3.1. A CH system is a quadruple (H, B, F, W) , where the function $H : T^*Q \rightarrow \mathbb{R}$ is called the Hamiltonian, $B \in \Gamma(\wedge^2 TT^*Q)$ is called an almost Poisson tensor (the main point being that it need not satisfy the Jacobi identity) on T^*Q , the fiber-preserving map $F : T^*Q \rightarrow T^*Q$ is called the (external) force map, and the subbundle W of T^*Q is called the control subbundle. When a feedback control law $u : T^*Q \rightarrow W$ is chosen, the quadruple (H, B, F, u) denotes the closed-loop CH system.

We remark that we choose to use almost Poisson tensors rather than almost symplectic forms, as it is more general and, moreover, is convenient in performing Poisson reduction. The vector field $X_{(H,B,F,u)}$ of a CH system (H, B, F, W) with a control law u is given by

$$X_{(H,B,F,u)} = B^\sharp \mathbf{d}H + \text{vlift}(F) + \text{vlift}(u),$$

where $\text{vlift}(F)$ and $\text{vlift}(u)$ are the vertical lifts defined in (2.1).

Reduction of G -invariant CH systems. We define G -invariant CH systems on T^*Q and reduced CH systems on T^*Q/G as follows.

DEFINITION 3.2. Let G be a Lie group action on Q . A G -invariant CH (G -CH) system is a CH system (H, B, F, W) , where H , B , F , and W are all G -invariant.

DEFINITION 3.3. A reduced CH (RCH) system is a quadruple (h, b, f, U) , where the function $h : T^*Q/G \rightarrow \mathbb{R}$ is called the reduced Hamiltonian, $b \in \Gamma(\wedge^2 T(T^*Q/G))$ is called the reduced almost Poisson tensor, the fiber-preserving map $f : T^*Q/G \rightarrow T^*Q/G$ is called the reduced force map, and the subbundle U of T^*Q/G is called the reduced control subbundle.

Suppose that we are given a G -CH system (H, B, F, W) on T^*Q . The G -invariant Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$ induces the reduced Hamiltonian $h : T^*Q/G \rightarrow \mathbb{R}$ as follows:

$$(3.1) \quad H = h \circ \pi_{/G}.$$

The G -invariance of the almost Poisson tensor $B \in \Gamma(\wedge^2 TT^*Q)$ induces a reduced almost Poisson tensor $[B]_G \in \Gamma(\wedge^2 T(T^*Q/G))$ as follows: for $f_1, f_2 \in \mathcal{F}(T^*Q/G)$,

$$(3.2) \quad [B]_{G[q,p]_G}(\mathbf{d}f_1, \mathbf{d}f_2) = B_{(q,p)}(\pi_{/G}^* \mathbf{d}f_1, \pi_{/G}^* \mathbf{d}f_2).$$

This is well defined since

$$\begin{aligned} B_{g(q,p)}(\mathbf{d}(f_1 \circ \pi_{/G}), \mathbf{d}(f_2 \circ \pi_{/G})) &= B_{(q,p)}(g^* \mathbf{d}(f_1 \circ \pi_{/G}), g^* \mathbf{d}(f_2 \circ \pi_{/G})) \\ &= B_{(q,p)}(\mathbf{d}(f_1 \circ \pi_{/G} \circ g), \mathbf{d}(f_2 \circ \pi_{/G} \circ g)) \\ &= B_{(q,p)}(\mathbf{d}(f_1 \circ \pi_{/G}), \mathbf{d}(f_2 \circ \pi_{/G})) \end{aligned}$$

for any $g \in G$, where we used the G -invariance of B in the first equality. One can easily check that $[B]_G$ is skew-symmetric. The G -invariance of F induces the reduced

force $[F]_G : T^*Q/G \rightarrow T^*Q/G$ satisfying

$$(3.3) \quad [F]_G \circ \pi_{/G} = \pi_{/G} \circ F.$$

This discussion motivates the following definition.

DEFINITION 3.4. *The RCH system of a G -CH system (H, B, F, W) is a quadruple $(h, [B]_G, [F]_G, W/G)$, where h is the reduced Hamiltonian defined in (3.1), $[B]_G$ is the reduced almost Poisson tensor defined in (3.2), and $[F]_G$ is the reduced force defined in (3.3).*

Similarly to Proposition 2.5, the following proposition explains the relations between G -CH systems and RCH systems.

PROPOSITION 3.5. *Given a RCH system (h, b, f, U) , there is a (not necessarily unique) G -CH system (H, B, F, W) whose RCH system is (h, b, f, U) .*

Proof. Define H by $H = h \circ \pi_{/G}$. Define a force map F on T^*Q as follows: for $\alpha_q \in T_q^*Q$, $v_q \in T_qQ$

$$\langle F(\alpha_q), v_q \rangle = \langle f \circ \pi_{/G}(\alpha_q), \tau_{/G}(v_q) \rangle.$$

Choose a connection on the principal bundle $T^*Q \rightarrow T^*Q/G$. (See Chapter 2, Theorem 2.1 in [25] for the proof of the existence.) Then we can split TT^*Q into the vertical space V and the horizontal space H as $TT^*Q = V \oplus H$. This induces the decomposition of T^*T^*Q as $T^*T^*Q = H^\circ \oplus V^\circ$, where H° and V° are the annihilators of H and V , respectively. Let $\text{hor} : T(T^*Q/G) \rightarrow H$ be the horizontal lift. Then its dual map $\text{hor}^* : V^\circ \rightarrow T^*(T^*Q/G)$ is an isomorphism. For simplicity, we use H° (resp., V°) as the projection of T^*T^*Q onto H° (resp., V°). Define an almost Poisson tensor B on T^*Q as follows: for $\alpha, \beta \in T_p^*T^*Q$

$$B(\alpha, \beta) := b(\text{hor}^* V^\circ \alpha, \text{hor}^* V^\circ \beta).$$

One can check that this almost Poisson tensor is G -invariant. We now show that $\pi_{/G} : T^*Q \rightarrow T^*Q/G$ is the Poisson map; i.e., $b = [B]_G$. Let h_1, h_2 be two functions on T^*Q/G . Then $\mathbf{d}(h_i \circ \pi_{/G}) \in V^\circ$, $i = 1, 2$. Thus, $\text{hor}^* \mathbf{d}(h_i \circ \pi_{/G}) = \mathbf{d}h_i$, $i = 1, 2$. Hence,

$$\begin{aligned} [B]_G(\mathbf{d}h_1, \mathbf{d}h_2) &= B(\mathbf{d}(h_1 \circ \pi_{/G}), \mathbf{d}(h_2 \circ \pi_{/G})) \\ &= b(\mathbf{d}h_1, \mathbf{d}h_2). \end{aligned}$$

It follows that $[B]_G = b$. Let $W = \pi_{/G}^{-1}(U)$. Then one can see that (H, B, F, W) is a G -CH system and that its RCH system coincides with (h, b, f, U) . This completes the proof. \square

Notice in Proposition 3.5 that there can be more than one B satisfying $[B]_G = b$, which is the source of the nonuniqueness of the G -CH systems (H, B, F, W) in Proposition 3.5. By Proposition 3.5, we can write an arbitrary RCH system in the form of the RCH system of a G -CH system without loss of generality.

Reduced CH dynamics. Given a G -CH system (H, B, F, W) , let $(h, [B]_G, [F]_G, W/G)$ be its RCH system. The (reduced) Hamiltonian vector field $X_{(h, [B]_G, [F]_G, [u]_G)}$ of $(h, [B]_G, [F]_G, W/G)$ with a control $[u]_G \in W/G$ is given by

$$(3.4) \quad X_{(h, [B]_G, [F]_G, [u]_G)} = [B]_G^\sharp \mathbf{d}h + \text{vlift}([F]_G) + \text{vlift}([u]_G),$$

where $\text{vlift}([F]_G)$ and $\text{vlift}([u]_G)$ are the vertical lifts defined in (2.1). Let $X_{(H, B, F, u)}$ be the vector field of (H, B, F, W) with control $u \in W$. Then we have

$$(3.5) \quad X_{(h, [B]_G, [F]_G, [u]_G)} \circ \pi_{/G} = T\pi_{/G} \cdot X_{(H, B, F, u)}.$$

The CH-equivalence relation. We shall first recall the CH-equivalence relation among CH systems on T^*Q from [20].

DEFINITION 3.6. *Two CH systems (H_i, B_i, F_i, W_i) , $i = 1, 2$, are said to be CH-equivalent, or simply, $(H_1, B_1, F_1, W_1) \stackrel{H}{\sim} (H_2, B_2, F_2, W_2)$, if the following Hamiltonian matching conditions hold:*

$$\text{HM-1: } W_1 = W_2,$$

$$\text{HM-2: } \text{Im}[B_1^\sharp \mathbf{d}H_1 + \text{vlift}(F_1) - B_2^\sharp \mathbf{d}H_2 - \text{vlift}(F_2)] \subset \text{vlift}(W_1),$$

where $\text{vlift}(W_1)$ is the vertical lift of W_1 defined in (2.2).

The following proposition explains the significance of the CH-equivalence relation.

PROPOSITION 3.7. *Suppose that two CH systems (H_i, B_i, F_i, W_i) , $i = 1, 2$, are CH-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two control laws u_i , $i = 1, 2$, is given by*

$$\text{vlift}(u_1) = -B_1^\sharp \mathbf{d}H_1 - \text{vlift}(F_1) + B_2^\sharp \mathbf{d}H_2 + \text{vlift}(F_2) + \text{vlift}(u_2).$$

Proof. Just compare $X_{(H_1, B_1, F_1, u_1)}$ and $X_{(H_2, B_2, F_2, u_2)}$ with controls $u_1 : T^*Q \rightarrow W_1$ and $u_2 : T^*Q \rightarrow W_2$. \square

RCH-equivalence. We now introduce an equivalence relation among RCH systems on T^*Q/G as follows.

DEFINITION 3.8. *Two RCH systems, $(h_i, [B_i]_G, [F_i]_G, W_i/G)$, $i = 1, 2$, are said to be reduced-CH-equivalent (RCH-equivalent), or simply*

$$(h_1, [B_1]_G, [F_1]_G, W_1/G) \stackrel{H}{\sim} (h_2, [B_2]_G, [F_2]_G, W_2/G),$$

if the following reduced Hamiltonian matching conditions hold:

$$\text{RHM-1: } W_1/G = W_2/G,$$

$$\text{RHM-2: } \text{Im}[[B_1]_G^\sharp \mathbf{d}h_1 + \text{vlift}([F_1]_G) - [B_2]_G^\sharp \mathbf{d}h_2 - \text{vlift}([F_2]_G)] \subset \text{vlift}(W_1/G),$$

where $\text{vlift}(W_1/G)$ is the vertical lift of the subbundle W_1/G defined in (2.2).

Reduction commutes with equivalence. The following theorem explains the relation between the RCH-equivalence relation among RCH systems on T^*Q/G and the CH-equivalence relation among G -CH systems on T^*Q .

THEOREM 3.9. *Two G -CH systems are CH-equivalent if and only if their associated RCH systems are RCH-equivalent.*

Proof. Use Definitions 3.6 and 3.8 as well as the relation (3.5). \square

THEOREM 3.10. *Suppose that two RCH systems $(h_i, [B_i]_G, [F_i]_G, W_i/G)$, $i = 1, 2$, are RCH-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop RCH systems produce the same equations of motion. The explicit relation between the two control laws $[u_i]_G$, $i = 1, 2$, is given by*

$$\text{vlift}([u_1]_G) = -[B_1]_G^\sharp \mathbf{d}h_1 - \text{vlift}([F_1]_G) + [B_2]_G^\sharp \mathbf{d}h_2 + \text{vlift}([F_2]_G) + \text{vlift}([u_2]_G).$$

Proof. The proof follows from a straightforward computation using (3.4). \square

Simple CH systems. Let us review the definition of simple CH systems from [20].

DEFINITION 3.11. *A CH system (H, B, F, W) on T^*Q is called simple if the Hamiltonian H has the form kinetic plus potential energy,*

$$H(q, p) = \frac{1}{2} \langle p, m_q^{-1} p \rangle + V(q),$$

and its almost Poisson tensor B is nondegenerate and is of the form

$$(3.6) \quad B(q, p) = \begin{bmatrix} 0 & K(q)^T \\ -K(q) & J(q, p) \end{bmatrix}$$

in cotangent coordinates (q, p) on T^*Q , where $K(q), J(q, p)$ are $n \times n$ matrices with $n = \dim Q$. We call H the simple Hamiltonian and B the simple almost Poisson tensor. The acronym, *SCH*, denotes “simple controlled Hamiltonian.”

One can check that the statement that B has the form (3.6) is independent of the choice of cotangent bundle coordinates on T^*Q .

Let B be a simple almost Poisson tensor. Then the relation

$$(3.7) \quad \text{vlift}(\psi_B) = B \circ \Theta$$

defines a unique $\psi_B \in \Gamma(\text{Aut}(T^*Q))$, where Θ is the canonical 1-form¹ on T^*Q and B is regarded as a bundle map $B : T^*T^*Q \rightarrow TT^*Q$. In other words, there exists a unique $\psi_B \in \Gamma(\text{Aut}(T^*Q))$ such that the following diagram commutes:

$$\begin{array}{ccc} T^*T^*Q & \xrightarrow{B} & TT^*Q \\ \uparrow \Theta & & \uparrow \text{vlift} \\ T^*Q & \xrightarrow{\psi_B} & T^*Q \end{array}$$

See [20] for the proof of the existence of ψ_B . When B is given in coordinates as in (3.6), the map ψ_B is given in coordinates by

$$\psi_B(q, p) = (q, K(q)p).$$

Now, we make the following definition of the reduced simple CH system.

DEFINITION 3.12. *An RCH system $(h, [B]_G, [F]_G, W/G)$ is called a reduced simple CH system (or RSCH system) if it is the reduced CH system of a G -invariant simple CH system.*

Simple almost Poisson tensors and their reductions. Recall that in Definition 3.11 we defined simple almost Poisson tensors on T^*Q using local coordinates. Here, we characterize the reduced simple almost Poisson tensors using local coordinates, so we may assume that $Q = G \times X$, where G is a Lie group acting on the manifold X trivially. Recall the following identifications by left translation of G :

$$T^*G = G \times \mathfrak{g}^*, \quad TT^*G = (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*), \quad T^*T^*G = (G \times \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g}).$$

We use (g_a, μ_a, x_i, p_i) as local coordinates for $T^*Q = G \times \mathfrak{g}^* \times T^*X$, and (μ_a, x_i, p_i) for $T^*Q/G = \mathfrak{g}^* \times T^*X$, where $a = 1, \dots, \dim G$ and $i = 1, \dots, \dim X$. We will use $\{e_a\}$ as a basis for \mathfrak{g} , and $\{e_a^*\}$ as its dual basis. Let $B \in \Gamma(\wedge^2 TT^*Q)$ be a G -invariant simple almost Poisson tensor. Then it is of the following form:

$$(3.8) \quad \begin{aligned} B(g, \mu, x, p) = & A_{ab}(x)(e_a \otimes e_b^* - e_b^* \otimes e_a) + E_{ia}(x)(\partial_{x_i} \otimes e_b^* - e_b^* \otimes \partial_{x_i}) \\ & + C_{ai}(x)(e_a \otimes \partial_{p_i} - \partial_{p_i} \otimes e_a) + D_{ij}(x)(\partial_{x_i} \otimes \partial_{p_j} - \partial_{p_j} \otimes \partial_{x_i}) \\ & + R_{ab}(\mu, x, p)e_a^* \otimes e_b^* + S_{ai}(\mu, x, p)(e_a^* \otimes \partial_{p_i} - \partial_{p_i} \otimes e_a^*) \\ & + U_{ij}(\mu, x, p)\partial_{p_i} \otimes \partial_{p_j}. \end{aligned}$$

¹The canonical 1-form Θ on T^*Q is given by $\Theta = p_i dq^i$ in cotangent bundle coordinates $(q^i, p_j) \in T^*Q$.

In the matrix form, B is given by

$$B = \begin{bmatrix} O & O & A(x) & C(x) \\ O & O & E(x) & D(x) \\ -A(x)^T & -E(x)^T & R(\mu, x, p) & S(\mu, x, p) \\ -C(x)^T & -D(x)^T & -S(\mu, x, p)^T & U(\mu, x, p) \end{bmatrix},$$

where we used the basis for $T_z T^*Q$ in the following order: $e_a, \partial_{x_i}, e_a^*, \partial_{p_i}$. The nondegeneracy condition for B is given by

$$\text{rank} \begin{bmatrix} A & C \\ E & D \end{bmatrix} = \dim Q.$$

The reduced simple Poisson tensor $[B]_G$ is given by

$$\begin{aligned} [B]_G(\mu, x, p) = & E_{ia}(x)(\partial_{x_i} \otimes e_b^* - e_b^* \otimes \partial_{x_i}) + D_{ij}(x)(\partial_{x_i} \otimes \partial_{p_j} - \partial_{p_j} \otimes \partial_{x_i}) \\ & + R_{ab}(\mu, x, p)e_a^* \otimes e_b^* + S_{ai}(\mu, x, p)(e_a^* \otimes \partial_{p_i} - \partial_{p_i} \otimes e_a^*) \\ & + U_{ij}(\mu, x, p)\partial_{p_i} \otimes \partial_{p_j}. \end{aligned} \quad (3.9)$$

In a matrix form,

$$[B]_G = \begin{bmatrix} R(\mu, x, p) & -E(x)^T & S(\mu, x, p) \\ E(x) & O & D(x) \\ -S(\mu, x, p)^T & -D(x)^T & U(\mu, x, p) \end{bmatrix},$$

where we used the basis for $T_{[z]}(T^*Q/G)$ in the following order: $e_a^*, \partial_{x_i}, \partial_{p_i}$. The nondegeneracy condition for B induces the following rank condition for $[B]_G$:

$$\text{rank} \begin{bmatrix} E & D \end{bmatrix} = \dim X. \quad (3.10)$$

Remark. One could argue that it might be better if we could characterize all the tensors $b \in \Gamma(\wedge^2 T(T^*Q/G))$ for which there exists a G -invariant simple almost Poisson tensor B such that $b = [B]_G$. Then we could define RSCH systems without reference to G -invariant SCH systems. This point has to be studied more, and we think that the use of connections is crucial; see [31] and [17].

4. Equivalence between reduced simple Lagrangian and Hamiltonian systems. In this section we show the equivalence between the method of RSCL systems on TQ/G and that of RSCH systems on T^*Q/G .

The CL-CH equivalence theorem. In [19] and [20], the equivalence between the method of SCL systems on TQ and that of SCH systems on T^*Q was shown. It is summarized in the following theorem.²

THEOREM 4.1. *The method of CL systems is equivalent to that of CH systems for simple mechanical systems in the following sense:*

1. *For any two simple CL systems (L_i, F_i^L, W_i^L) , $i = 1, 2$, there exist two associated simple CH systems³ (H_i, B_i, F_i^H, W_i^H) , $i = 1, 2$, such that*

$$\begin{aligned} (L_1, F_1^L, W_1^L) &\stackrel{L}{\sim} (L_2, F_2^L, W_2^L) \\ \iff (H_1, B_1, F_1^H, W_1^H) &\stackrel{H}{\sim} (H_2, B_2, F_2^H, W_2^H). \end{aligned} \quad (4.1)$$

²The statement of Theorem 4.1 is slightly different from that of Corollary 4.1 in [20] in that there appears an additional term, $\psi_{B_2} \circ \psi_{B_1}^{-1} = m_{H_2}(m_{H_1})^{-1}$, in statement 2. However, one can still prove Theorem 4.1 of this paper using section 4 of [20]. For example, see [19] for the proof.

³Refer to section 4 of [20] to learn how to find the associated CH systems.

2. For any two simple CH systems (H_i, B_i, F_i^H, W_i^H) , $i = 1, 2$, there exist two associated simple CL systems (L_i, F_i^L, W_i^L) , $i = 1, 2$, such that

$$(H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H) \\ \iff (L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L) \quad \text{and} \quad \psi_{B_2} \circ \psi_{B_1}^{-1} = m_{H_2}(m_{H_1})^{-1},$$

with m_{H_i} the mass tensor of H_i and ψ_{B_i} defined in (3.7) for $i = 1, 2$.

In statement 1, one does not need the condition $\psi_{B_2} \circ \psi_{B_1}^{-1} = m_{H_2}(m_{H_1})^{-1}$ along with the CL equivalence condition for the two CL systems, because it automatically holds for the associated CH systems.

This result implies the following: suppose that we want to find all SCL systems which are equivalent to a given SCL system. One can directly search for them on the Lagrangian side using the CL equivalence relation. Alternatively, one can first find an SCH system which is associated with the given SCL system, secondly search for all SCH systems which are CH-equivalent to this SCH system, and finally transform those SCH systems to SCL systems. Those SCL systems are all SCL systems CL-equivalent to the original SCL system. In the similar way, one can find all the SCH systems which are CH-equivalent to a given SCH system, directly or with a CL-equivalence relation. (Refer to [20] and [19] for more detail.) Hence, one can describe a given simple mechanical system as an SCL system or as an SCH system and then apply the CL method or the CH method, correspondingly. Both procedures are equivalent.

We now restrict Theorem 4.1 to G -invariant systems.

THEOREM 4.2. *The method of G -invariant SCL systems and that of G -invariant SCH systems are equivalent. In other words, Theorem 4.1 restricted to G -invariant simple CL and CH systems holds.*

Proof. For the proof, one just needs to keep track of the G -invariance in the proof of Theorem 4.1 in [20] and the proof of Theorem 3.2.1 in [19]. \square

The following theorem explains the equivalence between the method of RSCL systems on TQ/G and that of RSCH systems on T^*Q/G .

THEOREM 4.3. *The method of RSCL systems is equivalent to that of RSCH systems in the following sense:*

1. For two given RSCL systems $(l_i, [F_i^L]_G, W_i^L/G)$, $i = 1, 2$, there exist two associated RSCH systems $(h_i, [B_i]_G, [F_i^H]_G, W_i^H/G)$, $i = 1, 2$, such that

$$(l_1, [F_1^L]_G, W_1^L/G) \overset{L}{\sim} (l_2, [F_2^L]_G, W_2^L/G) \\ (4.2) \quad \iff (h_1, [B_1]_G, [F_1^H]_G, W_1^H/G) \overset{H}{\sim} (h_2, [B_2]_G, [F_2^H]_G, W_2^H/G).$$

2. For two given RSCH systems $(h_i, [B_i]_G, [F_i^H]_G, W_i^H/G)$, $i = 1, 2$, there exist two associated RSCL systems $(l_i, [F_i^L]_G, W_i^L/G)$, $i = 1, 2$, such that

$$(h_1, [B_1]_G, [F_1^H]_G, W_1^H/G) \overset{H}{\sim} (h_2, [B_2]_G, [F_2^H]_G, W_2^H/G) \\ \iff (l_1, [F_1^L]_G, W_1^L/G) \overset{L}{\sim} (l_2, [F_2^L]_G, W_2^L/G) \quad \text{and} \quad [\psi_{B_2} \circ \psi_{B_1}^{-1}]_G = [m_{H_2}]_G [m_{H_1}]_G^{-1},$$

where $[m_{H_i}]_G$ is the reduced mass tensor of h_i .

Proof. Let us prove statement 1. For given two RSCL systems $(l_i, [F_i^L]_G, W_i^L/G)$, $i = 1, 2$, consider their unreduced G -SCL systems (L_i, F_i^L, W_i^L) , $i = 1, 2$, with $L_i = l_i \circ \tau_{/G}$ (see Proposition 2.5). Theorem 4.2 implies that there are two G -SCH systems

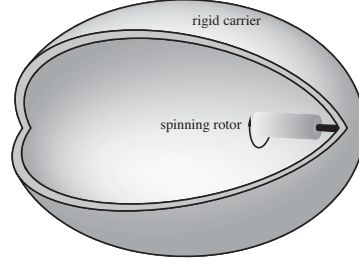


FIG. 5.1. A satellite with a rotor along the third body axis.

(H_i, B_i, F_i^H, W_i^H) , $i = 1, 2$, such that (4.1) holds. Let $(h_i, [B_i]_G, [F_i^H]_G, W_i^H/G)$ be the RSCH system of (H_i, B_i, F_i^H, W_i^H) for $i = 1, 2$. Then (4.2) follows from Theorems 2.10 and 3.9 and (4.1). Now we prove statement 2. In this case, use Proposition 3.5 instead of Proposition 2.5, and then proceed in a similar manner. \square

Hence, one can describe a given simple mechanical system as an SCL/SCH system, apply the CL/CH reduction, and then apply the reduced CL/CH method, correspondingly. Both procedures are equivalent.

Remark. Notice in (4.2) that $\psi_{B_2} \circ \psi_{B_1}^{-1}$ is G -equivariant even though each of ψ_{B_i} may not be. This equivariance is a consequence of the following commutative diagram:

$$\begin{array}{ccccc}
 TT^*Q & \xleftarrow{B_1} & T^*Q^*Q & \xrightarrow{B_2} & TT^*Q \\
 \uparrow \text{vlift} & & \uparrow \Theta & & \uparrow \text{vlift} \\
 T^*Q & \xleftarrow{\psi_{B_1}} & T^*Q & \xrightarrow{\psi_{B_2}} & T^*Q
 \end{array}$$

It follows that for $\alpha \in T^*Q$, $B_2 B_1^{-1}(\text{vlift}(\alpha)) = \text{vlift}(\psi_{B_2} \circ \psi_{B_1}^{-1}(\alpha))$. One can easily check that vlift is G -equivariant, i.e., $\text{vlift}(g\alpha) = g \text{vlift}(\alpha)$ for $g \in G$. Then the G -equivariance of $\psi_{B_2} \circ \psi_{B_1}^{-1}$ follows from the G -equivariance of B_1 , B_2 , and vlift and from the injectivity of vlift .

5. Examples. We review the example of a satellite with a rotor and the Euler–Poincaré matching conditions presented in [15] in the framework of the current paper. We then apply the RCL method to the stabilization of the heavy top with rotors. This will show the application of CL systems only. For an excellent application of CH systems to the problem of underwater vehicle stabilization by internal rotors, refer to [34] and [15].

5.1. Satellite with a rotor and Euler–Poincaré matching.

Satellite with a rotor (see [15]). We consider the example of a satellite with a rotor aligned along the third principal axis of the body (see Figure 5.1). The configuration space is $Q = G \times X = \text{SO}(3) \times S^1$, with the first factor being the satellite attitude and the second factor being the rotor angle. The Lie group $G = \text{SO}(3)$ acts on the first factor of Q only. We take a trivial (flat) connection on Q such that $TQ/G \simeq \mathfrak{g} \times TX$. Use $((\Omega_1, \Omega_2, \Omega_3), (\phi, \dot{\phi}))$ as coordinates for $\mathfrak{so}(3) \times TS^1 \simeq \mathbb{R}^3 \times TS^1$. This system is described by the RSCL system $(l_1, [F_1]_G = 0, W_1/G)$ given by

$$l_1(\Omega, \dot{\phi}) = \frac{1}{2}(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + (I_3 + J_3) \Omega_3^2 + 2J_3 \Omega_3 \dot{\phi} + J_3 \dot{\phi}^2)$$

and $W_1/G = \text{span}\{\mathbf{d}\phi\}$, where $\lambda_1 > \lambda_2 > \lambda_3 := J_3 + I_3$. Notice that l_1 does not depend on ϕ . Recall that the reduced Euler–Lagrange operator \mathcal{REL} induces the Lagrange–Poincaré operator \mathcal{LP} in (2.8) with respect to the trivial connection on $Q \rightarrow X$. By (2.10), this Lagrange–Poincaré operator $\mathcal{LP}(l_1)$ is given by

$$(5.1) \quad \mathcal{LP}(l_1) = \begin{bmatrix} \frac{d}{dt} \frac{\partial l_1}{\partial \Omega} - \frac{\partial l_1}{\partial \Omega} \times \Omega \\ \frac{d}{dt} \frac{\partial l_1}{\partial \dot{\phi}} - \frac{\partial l_1}{\partial \phi} \end{bmatrix},$$

where we switched the first and second components of (2.10).

Consider another RSCL system $(l_2, 0, W_2/G)$ with $W_2 = W_1$ and

$$l_2(\Omega, \dot{\phi}) = \frac{1}{2}(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + (I_3 + J_3) \Omega_3^2 + 2J_3 \Omega_3 \dot{\phi} + \rho J_3 \dot{\phi}^2)$$

with $\rho \in \mathbb{R}$. Here we allow only one free parameter ρ as in [15], but one can consider a more general form of CL system. One can check that RELM-1 and RELM-2 in Definition 2.9 are satisfied. Since these two CL systems are equivalent, one has only to design a controller for the second system, which will give an asymptotically stabilizing controller for the first system by Theorem 2.11. See [8] for the discussion on asymptotic stabilization of the rotation about the middle axis in the body-fixed frame. There, it is shown how to choose ρ and the dissipative input for stability of the equivalent system $(l_2, 0, W_2/G)$. This leads to an asymptotically stabilizing controller for the original system $(l_1, 0, W_1/G)$.

Euler–Poincaré matching. Here we briefly sketch the proof that the set of Euler–Poincaré matching conditions in [12] and [15] is a special case of the reduced Euler–Lagrange matching conditions in this paper. This set of matching conditions can handle such examples as a satellite with a rotor and underwater vehicles with internal rotors. Let $Q = G \times X$ be the configuration space, where G is a Lie group acting trivially on the manifold X . We choose the trivial (flat) connection on $Q \rightarrow X$ to write down the Lagrange–Poincaré equation on $TQ/G \simeq \mathfrak{g} \times TX$ with the Lie algebra \mathfrak{g} of the Lie group G . We use $\eta = (\eta^\alpha)$ as coordinates for \mathfrak{g} , and $(\theta, \dot{\theta}) = (\theta^a, \dot{\theta}^a)$ as coordinates for TX . By (2.10), the Lagrange–Poincaré operator \mathcal{LP} with respect to the *trivial* connection is given by

$$(5.2) \quad \mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt} \frac{\partial l}{\partial \eta^\alpha} - C_{\gamma\alpha}^\beta \eta^\gamma \frac{\partial l}{\partial \eta^\beta} \\ \frac{d}{dt} \frac{\partial l}{\partial \dot{\theta}^a} - \frac{\partial l}{\partial \theta^a} \end{pmatrix}$$

for any reduced Lagrangian $l = l(\eta^\alpha, \dot{\theta}^a, \theta)$, where $C_{\gamma\alpha}^\beta$ are the structure constants of the Lie algebra \mathfrak{g} . In (5.2) we wrote the vertical part of $\mathcal{LP}(l)$ first, while in (2.10) the vertical part was written in the second component.

Let $(l, 0, T^*X)$ be the given RSCL system with the reduced Lagrangian

$$l(\eta^\alpha, \dot{\theta}^a) = \frac{1}{2} g_{\alpha\beta} \eta^\alpha \eta^\beta + g_{\alpha a} \eta^\alpha \dot{\theta}^a + \frac{1}{2} g_{ab} \dot{\theta}^a \dot{\theta}^b,$$

where $g_{\alpha\beta}, g_{\alpha a}, g_{ab}$ are constant functions on TQ/G . Notice that this Lagrangian is cyclic in the variables θ^a and that the controls act only on the cyclic variables. Let

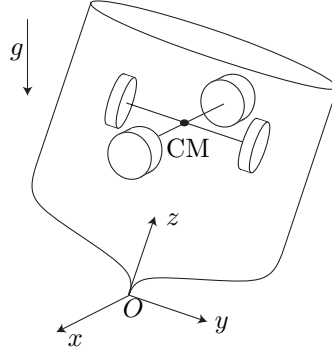


FIG. 5.2. Heavy top with two rotors, each consisting of two rigidly coupled disks. The center of mass is at CM.

$(l_{\tau,\sigma,\rho}, 0, T^*X)$ be another RSCL system with the reduced Lagrangian of the following form:

$$(5.3) \quad \begin{aligned} l_{\tau,\sigma,\rho} = & l(\eta^\alpha, \dot{\theta}^a + \tau_\alpha^a \eta^\alpha) + \frac{1}{2} \sigma_{ab} \tau_\alpha^a \tau_\beta^b \eta^\alpha \eta^\beta \\ & + \frac{1}{2} (\rho_{ab} - g_{ab}) (\dot{\theta}^a + g^{ac} g_{c\alpha} \eta^\alpha + \tau_\alpha^a \eta^\alpha) (\dot{\theta}^b + g^{bc} g_{c\beta} \eta^\beta + \tau_\beta^b \eta^\beta), \end{aligned}$$

which is exactly the equation (11) in [15]. See also [14] for the motivation of this choice of the form in (5.3). [15] proposes the following so-called Euler–Poincaré matching conditions:

$$\begin{aligned} \text{EP-1: } & \tau_\alpha^a = -\sigma^{ab} g_{b\alpha}, \\ \text{EP-2: } & \sigma^{ab} + \rho^{ab} = g^{ab}. \end{aligned}$$

Then one can show that the two assumptions of EP-1 and EP-2 imply the RCL-equivalence of the two RSCL systems $(l, 0, T^*X)$ and $(l_{\tau,\sigma,\rho}, 0, T^*X)$. Hence, one can equivalently work with the second system to design controllers. Refer to [15] for the method of constructing a Lyapunov function using the energy and Casimir functions.

5.2. Heavy top with rotors. It is well known that an upright spinning top is unstable if the angular velocity is small. The motion of a heavy top and the stability of the Lagrange top are well studied in [28] and [24]. We use the CL method to asymptotically stabilize the upright spinning motion of a heavy top with small vertical angular velocity, including zero velocity. See Figure 5.2 for the heavy top system. One can notice that the system dynamics are not $\text{SO}(3)$ -invariant because the gravitational force breaks the $\text{SO}(3)$ symmetry, and thus we cannot perform the usual reduction of the system by the $\text{SO}(3)$ group. However, there is a way of doing the $\text{SO}(3)$ -reduction of this system by considering this system as one depending on a parameter in \mathbb{R}^3 .

In this section, we will first review the general theory of reduction of systems depending on a parameter and then apply this reduction theory to the design of a controller for the heavy top system. We will not develop the whole theory of *CL systems depending on a parameter* and the reduction theory for those systems with symmetry because it is a straightforward modification of the theory in section 2 of this paper. Moreover, the complete theory of reduction for (uncontrolled) systems depending on a parameter is in [17].

Systems depending on a parameter. We here review the reduction theory for systems depending on an advected parameter, as presented in [17]. Consider a Lagrangian

$$L : T(G \times X) \times V^* \rightarrow \mathbb{R},$$

where G is a Lie group, X is a manifold, and V^* is the dual space of the vector space V . The value of L at the point $(g, x, \dot{g}, \dot{x}, a_0) \in T(G \times X) \times V^*$ will be denoted by $L(g, x, \dot{g}, \dot{x}, a_0)$, as usual, and we will think of a_0 as a parameter that remains fixed along the evolution of the system. Assume that there is an action of G on V , so there is an induced action on V^* such that $\langle ga_0, gb_0 \rangle = \langle a_0, b_0 \rangle$ for all $a_0 \in V^*$, $b_0 \in V$, $g \in G$. Assume that G acts trivially on X and that L is G -invariant, i.e.,

$$L(hg, x, h\dot{g}, \dot{x}, ha_0) = L(g, x, \dot{g}, \dot{x}, a_0)$$

for $h \in G$, $(g, x, \dot{g}, \dot{x}, a_0) \in T(G \times X) \times V^*$. Define the reduced Lagrangian $l : \mathfrak{g} \times TX \times V^* \rightarrow \mathbb{R}$ by

$$(5.4) \quad l(\xi, x, \dot{x}, a) = L(e, x, \xi, \dot{x}, a)$$

for $\xi \in \mathfrak{g}$, $(x, \dot{x}) \in TX$, $a \in V^*$, where $\xi = g^{-1}\dot{g}$ and $a = g^{-1}a_0$.

Fix $a_0 \in V^*$. Let $L_{a_0} : T(G \times X) \rightarrow \mathbb{R}$ be the restriction of L to $T(G \times X) \times \{a_0\}$. Then, the Euler–Lagrange operator $\mathcal{EL}(L_{a_0})$ induces the reduced Euler–Lagrange operator

$$\mathcal{REL}(l) : 2\tilde{\mathfrak{g}} \oplus T^{(2)}X \times TV^* \rightarrow \mathfrak{g}^* \times T^*X$$

and the equation in (5.6) as follows:

$$(5.5) \quad \mathcal{REL}(l) = \begin{pmatrix} \frac{d}{dt} \frac{\partial l}{\partial \xi} - \text{ad}_\xi^* \frac{\partial l}{\partial \xi} - \frac{\partial l}{\partial a} \diamond a \\ \frac{d}{dt} \frac{\partial l}{\partial \dot{x}} - \frac{\partial l}{\partial x} \end{pmatrix}$$

and

$$(5.6) \quad \dot{a} = -\xi a,$$

where the map $\diamond : V \times V^* \rightarrow \mathfrak{g}^*$ is defined by $\langle b \diamond a, \eta \rangle = -\langle \eta a, b \rangle$ for $\eta \in \mathfrak{g}$, $a \in V^*$, $b \in V$. One may find the derivation of (5.5) and (5.6) in section 7.4 of [17].

The equations of motion of the reduced Lagrangian l with a $(\mathfrak{g}^* \times T^*X)$ -valued (reduced) force f , are given by

$$(5.7) \quad \mathcal{REL}(l) = f \quad \text{and} \quad \dot{a} = -\xi a,$$

where f includes external forces and control forces.

Heavy top with two pairs of rotors. We first describe a heavy top with two pairs of rotors. We mount two pairs of rotors within the top so that each pair's rotation axis is parallel to the first and the second principal axes of the top; see Figure 5.2. Let I_1, I_2, I_3 be the moments of inertia of the top in the body-fixed frame. Let J_1, J_2 be the moments of inertia of the rotors around their rotation axes. Let J_{i1}, J_{i2}, J_{i3} be the moments of inertia of the i th rotor with $i = 1, 2$ around the first, the second, and the third principal axis, respectively. Let $\bar{I}_1 = I_1 + J_{11} + J_{21}$, $\bar{I}_2 = I_2 + J_{12} + J_{22}$,

and $\bar{I}_3 = I_3 + J_{13} + J_{23}$. Let $\lambda_1 = \bar{I}_1 + J_1$ and $\lambda_2 = \bar{I}_2 + J_2$. Let M be the total mass of the system, g the magnitude of the gravitational acceleration, and h the distance from the origin O to the center of mass of the system.

Let $G = \text{SO}(3)$, $X = S^1 \times S^1$, $V^* = \mathbb{R}^3$. We use the following notation for coordinates:

$$\mathbf{R} \in \text{SO}(3), \quad \theta = (\theta_1, \theta_2) \in S^1 \times S^1, \quad a \in \mathbb{R}^3.$$

We will use $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ as coordinates for the Lie algebra $\mathfrak{so}(3)$ under the Lie algebra isomorphism, $\vee : (\mathfrak{so}(3), [\cdot, \cdot]) \rightarrow (\mathbb{R}^3, \times)$,

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}^\vee = (x, y, z).$$

Let $L : T(G \times X) \times V^* \rightarrow \mathbb{R}$ be the Lagrangian defined by

$$L(\mathbf{R}, \theta, \dot{\mathbf{R}}, \dot{\theta}, a) = \frac{1}{2} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 & 0 & J_1 & 0 \\ 0 & \lambda_2 & 0 & 0 & J_2 \\ 0 & 0 & \bar{I}_3 & 0 & 0 \\ J_1 & 0 & 0 & J_1 & 0 \\ 0 & J_2 & 0 & 0 & J_2 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} - Mgh\mathbf{R}^{-1}a \cdot \chi,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3) = (\mathbf{R}^{-1}\dot{\mathbf{R}})^\vee$ is the body-fixed angular momentum and χ is the body-fixed unit vector on the line segment connecting the origin O with the body's center of mass, i.e., $\chi = (0, 0, 1)$ in the body-fixed frame.

Fix $\mathbf{k} := (0, 0, 1) \in \mathbb{R}^3$. Let $L_{\mathbf{k}}$ be the restriction of L to $T(G \times X) \times \{\mathbf{k}\}$ as follows:

$$L_{\mathbf{k}}(\mathbf{R}, \theta, \dot{\mathbf{R}}, \dot{\theta}) = L(\mathbf{R}, \theta, \dot{\mathbf{R}}, \dot{\theta}, \mathbf{k}).$$

Then one can check that $L_{\mathbf{k}}$ is the Lagrangian of the heavy top system in Figure 5.2. The actuation is exerted on each pair of rotors, so the control bundle U is given by $U = T^*X$.

By (5.4), the reduced Lagrangian $l : \mathfrak{so}(3) \times TX \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} l(\Omega, \theta, \dot{\theta}, \Gamma) &= \frac{1}{2}(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + \bar{I}_3\Omega_3^2 + 2J_1\Omega_1\dot{\theta}_1 + 2J_2\Omega_2\dot{\theta}_2) \\ &\quad + \frac{1}{2}(J_1\dot{\theta}_1^2 + J_2\dot{\theta}_2^2) - Mgh\Gamma \cdot \chi, \end{aligned}$$

where $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = \mathbf{R}^{-1}\mathbf{k}$. Physically, the vector Γ represents the motion of the *unit vector* with the *opposite direction* of gravity as seen from the body. Recall that the reduced equations of motion are derived from (5.7) and (5.5).

Let us consider a new reduced Lagrangian \tilde{l} defined by

$$\begin{aligned} \tilde{l}(\Omega, \theta, \dot{\theta}, \Gamma) &= \frac{1}{2}(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + \bar{I}_3\Omega_3^2 + 2J_1\Omega_1\dot{\theta}_1 + 2J_2\Omega_2\dot{\theta}_2) \\ (5.8) \quad &\quad + \frac{1}{2}(\rho_1 J_1 \dot{\theta}_1^2 + \rho_2 J_2 \dot{\theta}_2^2) - Mgh\Gamma \cdot \chi, \end{aligned}$$

where $\rho_1, \rho_2 \in \mathbb{R}$ are free parameters to be chosen later. See [15] and [21] for the motivation of this choice of the form in (5.8).

Even though we have not developed the general theory of CL systems depending on a parameter and the associated reduction theory, one can check that Definition 2.9 and Theorem 2.11 hold for reduced CL systems depending on a parameter where one should use the reduced Euler–Lagrange operator in (5.5) instead of (2.9) or (2.10). Notice that the equation in (5.6) is common for all reduced CL systems depending on a parameter. With these modifications in mind, one can check that

$$(l, 0, T^*X) \stackrel{L}{\sim} (\tilde{l}, 0, T^*X).$$

By Theorem 2.11, we can work with the RCL system $(\tilde{l}, 0, T^*X)$ to design a controller.

Let us define the angular momentum, $\Pi = (\Pi_1, \Pi_2, \Pi_3)$, and the control momentum, $\tilde{J} = (\tilde{J}_1, \tilde{J}_2)$, as follows:

$$(5.9) \quad \Pi = \frac{\partial \tilde{l}}{\partial \Omega} = (\lambda_1 \Omega_1 + J_1 \dot{\theta}_1, \lambda_2 \Omega_2 + J_2 \dot{\theta}_2, \bar{I}_3 \Omega_3),$$

$$(5.10) \quad \tilde{J} = \frac{\partial \tilde{l}}{\partial \dot{\theta}} = (J_1 \Omega_1 + J_1 \rho_1 \dot{\theta}_1, J_2 \Omega_2 + J_2 \rho_2 \dot{\theta}_2).$$

By (5.7) and (5.5), the equations of motion of the system $(\tilde{l}, 0, T^*X)$ with a choice of control $v = (v_1, v_2)$ are given as follows:

$$(5.11) \quad \dot{\Pi} = \Pi \times \Omega + Mgh\Gamma \times \chi,$$

$$(5.12) \quad \dot{\tilde{J}} = v,$$

$$(5.13) \quad \dot{\Gamma} = \Gamma \times \Omega.$$

These dynamics have two constants of motion,

$$\Pi \cdot \Gamma \quad \text{and} \quad \|\Gamma\| = 1,$$

where $\Pi \cdot \Gamma$ is the vertical component of the space-fixed angular momentum, and $\|\Gamma\| = \|\mathbf{R}^{-1}\mathbf{k}\| = \|\mathbf{k}\| = 1$.

Since the reduced Lagrangian \tilde{l} (or l) does not depend on the rotor angle θ , and we are not interested in the angle of rotors but in the angular velocity of rotors, we will remove X from the *phase space* for the sake of simplicity. Hence, we will regard $\mathfrak{so}(3) \times \mathbb{R}^2 \times \mathbb{R}^3$ as a new phase space, where \mathbb{R}^2 is the velocity component of $TX = \mathbb{R}^2 \times \mathbb{R}^2$.

Let $\Omega(0)$, $\dot{\theta}(0)$, and $\Gamma(0)$ with $\|\Gamma(0)\|^2 = 1$ be an initial condition with

$$(5.14) \quad \Omega_3^\circ := \Pi(0) \cdot \frac{\Gamma(0)}{\bar{I}_3} < \sqrt{\frac{Mgh}{\bar{I}_3}}.$$

We are interested in the equilibrium $e = (\Omega_e, \dot{\theta}_e, \Gamma_e)$,

$$(5.15) \quad \Omega_e = (0, 0, \Omega_3^\circ), \quad \dot{\theta}_e = (0, 0), \quad \Gamma_e = (0, 0, 1)$$

or

$$\Omega_e = (0, 0, \Omega_3^\circ), \quad \tilde{J}_e = (0, 0), \quad \Gamma_e = (0, 0, 1),$$

which corresponds to the upright spinning top with the rotors at rest. Notice that this equilibrium lies in the same level set of $(\Pi \cdot \Gamma, \|\Gamma\|^2)$ as the initial condition.

We construct a Lyapunov function using the energy-Casimir method (see [8] for more detail of this method). Set

$$(5.16) \quad E_{\tilde{\Phi}} = K + U + \Phi(\Pi \cdot \Gamma, \|\Gamma\|^2) + \Psi(\tilde{J}_1, \tilde{J}_2),$$

where the potential energy U is given by $U(\Gamma) = Mgh\Gamma \cdot \chi = Mgh\Gamma_3$, and the kinetic energy K is given by

$$K = \frac{1}{2} \left(\lambda_1 - \frac{J_1}{\rho_1} \right) \Omega_1^2 + \frac{1}{2} \left(\lambda_2 - \frac{J_2}{\rho_2} \right) \Omega_2^2 + \frac{1}{2} \bar{I}_3 \Omega_3^2 + \frac{\tilde{J}_1^2}{2J_1\rho_1} + \frac{\tilde{J}_2^2}{2J_2\rho_2},$$

in the new coordinates $(\Omega, \tilde{J}, \Gamma)$. Choose the function Ψ as follows:

$$(5.17) \quad \Psi(\tilde{J}_1, \tilde{J}_2) = \frac{\tilde{J}_1^2}{2\epsilon_1 J_1} + \frac{\tilde{J}_2^2}{2\epsilon_2 J_2},$$

where coefficients ϵ_i will be determined later. Choose the function Φ of the form

$$\begin{aligned} \Phi(x, y) = & -\Omega_3^\circ(x - \bar{I}_3\Omega_3^\circ) + \frac{1}{2} (\bar{I}_3(\Omega_3^\circ)^2 - Mgh)(y - 1) \\ & + \frac{1}{2} a_1(x - \bar{I}_3\Omega_3^\circ)^2 + \frac{1}{2} a_2(y - 1)^2, \end{aligned}$$

where the constants a_1 and a_2 are chosen such that

$$a_1 < \frac{-1}{\bar{I}_3}$$

and

$$4a_2 + a_1(\bar{I}_3\Omega_3^\circ)^2 + \bar{I}_3(\Omega_3^\circ)^2 - Mgh < \frac{\bar{I}_3(a_1\bar{I}_3\Omega_3^\circ - \Omega_3^\circ)^2}{1 + a_1\bar{I}_3}.$$

One can check that the equilibrium e is a critical point of $E_{\tilde{\Phi}}$. We now find conditions under which this critical point is a local *maximum*. First, choose ρ_i satisfying

$$(5.18) \quad \frac{\bar{I}_3(\Omega_3^\circ)^2 - Mgh}{(\Omega_3^\circ)^2} < \lambda_i - \frac{J_i}{\rho_i} < 0$$

for $i = 1, 2$, and then we can choose ϵ_1 and ϵ_2 such that the second derivative of $E_{\tilde{\Phi}}$ becomes negative definite at e , which implies that $E_{\tilde{\Phi}}$ has a local maximum at e . For later use, we impose an additional condition on ρ_i and ϵ_i as follows:

$$(5.19) \quad J_i(\Omega_3^\circ)^2 + (\epsilon_i + \rho_i) ((\Omega_3^\circ)^2(\bar{I}_3 - \lambda_i) - Mgh) \neq 0.$$

With (5.19), it is still possible to find ρ_i and ϵ_i to ensure negative definiteness of the second derivative of $E_{\tilde{\Phi}}$ at e .

The following choice of $v = (v_1, v_2)$,

$$(5.20) \quad v_i = c_i \left(\dot{\theta}_i + \frac{\tilde{J}_i}{\epsilon_i J_i} \right),$$

with $c_i > 0$ for $i = 1, 2$, implies

$$(5.21) \quad \frac{d}{dt} E_{\tilde{\Phi}} = \sum_{i=1}^2 c_i \left(\dot{\theta}_i + \frac{\tilde{J}_i}{\epsilon_i J_i} \right)^2 \geq 0,$$

which proves the Lyapunov stability of the equilibrium e in the closed-loop system. The complete control law u for the original system $(l, 0, T^*X)$ can be obtained from Theorem 2.11.

Asymptotic stabilization will now be shown by using LaSalle's theorem. Since $E_{\tilde{\Phi}}$ has a local maximum at e , it is nondecreasing in time, and $\Pi \cdot \Gamma$ and $\|\Gamma\|^2$ are conserved, there is a number c such that the set

$$S = \{x \in \mathfrak{so}(3) \times \mathbb{R}^2 \times \mathbb{R}^3 \mid E_{\tilde{\Phi}} \geq c, \Pi \cdot \Gamma = \Pi_e \cdot \Gamma_e, \|\Gamma\|^2 = 1\}$$

is nonempty, compact, and positively invariant. Define

$$\mathcal{E} = \{x \in S \mid \dot{E}_{\tilde{\Phi}} = 0\} = \{x \in S \mid v = 0\}.$$

Let \mathcal{M} be the largest invariant subset of \mathcal{E} . One can show $\mathcal{M} = \{e\}$ by (5.19) after shrinking the set S if necessary. Thus, by LaSalle's theorem, e is asymptotically stable.

Here are the main points in the proof that $\mathcal{M} = \{e\}$. Let $(\Omega(t), \dot{\theta}(t), \Gamma(t))$ be a trajectory in \mathcal{M} . The condition $v = 0$ and (5.12) imply that $\tilde{J}(t)$ is constant. Hence, $\theta_i(t)$ and $\Omega_i(t)$ are constant for $i = 1, 2$. By (5.9), $\Pi_i(t)$, $i = 1, 2$ are constant. Then the third component of (5.11) becomes $\lambda_3 \dot{\Omega}_3(t) = \text{constant}$. By the Lyapunov stability of the equilibrium, it follows $\dot{\Omega}_3(t) \equiv 0$. Hence, $\Omega_3(t)$ is constant. The first and second component of (5.11) imply that $\Gamma_1(t)$ and $\Gamma_2(t)$ are constant. Then the third component of (5.13) implies that $\Gamma_3(t)$ is constant. So far we have shown that the trajectory $(\Omega(t), \dot{\theta}(t), \Gamma(t))$, or $(\Pi(t), \dot{\theta}(t), \Gamma(t))$, is constant for all $t \geq 0$. Consider the map $f : \mathbb{R}^8 \rightarrow \mathbb{R}^{10}$ defined by

$$f(\Omega, \dot{\theta}, \Gamma) = \begin{pmatrix} \Pi \times \Omega + Mgh\Gamma \times \chi \\ \Gamma \times \Omega \\ J_1\Omega_1 + (\epsilon_1 + J_1\rho_1)\dot{\theta}_1 \\ J_2\Omega_2 + (\epsilon_2 + J_2\rho_2)\dot{\theta}_2 \\ \Pi \cdot \Gamma - \Pi_e \cdot \Gamma_e \\ \|\Gamma\|^2 - 1 \end{pmatrix} = \begin{pmatrix} \dot{\Pi} \\ \dot{\Gamma} \\ \epsilon_1 v_1 \\ \epsilon_2 v_2 \\ \Pi \cdot \Gamma - \Pi_e \cdot \Gamma_e \\ \|\Gamma\|^2 - 1 \end{pmatrix},$$

where Π is expressed in terms of $(\Omega, \dot{\theta})$ as in (5.9). Then one can see that all the trajectories lying in \mathcal{M} are contained in the set $f^{-1}(O)$. In particular, the equilibrium $(\Omega_e, \dot{\theta}_e, \Gamma_e)$ in (5.15) is also contained in $f^{-1}(O)$. One can check that the rank of the Jacobian matrix Df at the equilibrium is the full rank 8 by (5.19). Thus, f is locally one-to-one around the equilibrium by Theorem 4.12 in [16]. Therefore, the only possible trajectory totally lying in \mathcal{M} is the equilibrium point itself. It follows from LaSalle's theorem that the equilibrium is asymptotically stable.

Remarks. 1. The above procedure shows that the choice of control gains depends on the initial condition. This is unavoidable because we need to know the value of the constant of motion $\Pi \cdot \Gamma$, which the internal actuation cannot change; however, our suggested controller is robust to small errors in the measurement of the initial condition. Let \tilde{e} be the equilibrium of the form (5.15), with $\tilde{\Omega}_3^\circ$ instead of Ω_3° . Suppose the Ω_3° used in constructing the control law is very close to the value $\tilde{\Omega}_3^\circ$. Let $\tilde{E}_{\tilde{\Phi}}$ be the function of the form (5.16), with Ω_3° replaced by $\tilde{\Omega}_3^\circ$. Then \tilde{e} is a critical point of $\tilde{E}_{\tilde{\Phi}}$. By continuity, the second derivative of $\tilde{E}_{\tilde{\Phi}}$ at \tilde{e} will remain negative definite, proving Lyapunov stability of \tilde{e} .

2. The same form of controller works for the asymptotic stabilization of the upright spinning top with $\Omega_3^\circ > \sqrt{Mgh/I_3}$, which is the opposite of (5.14). All

that needs to be done is to choose ρ_i and ϵ_i to make $E_{\tilde{\Phi}}$ have a local minimum at the equilibrium and to choose negative c_i such that $E_{\tilde{\Phi}}$ decreases in time. LaSalle's theorem argument guarantees asymptotic stability.

6. Conclusions. In this paper we have studied the reduction of controlled Lagrangian (CL) and controlled Hamiltonian (CH) systems with symmetry. We have shown that the notion of equivalence of controlled systems is preserved by the reduction procedure. This leads to a natural derivation of the Bloch–Leonard–Marsden Euler–Poincaré matching conditions and shows in a precise sense how they are related to the unreduced Euler–Lagrange matching conditions. The theory also shows how to do the equivalent matching on the Hamiltonian side. We studied the examples of a rigid body with rotors (a spacecraft) as well as a heavy top with rotors to illustrate the theory.

In the future we will study more examples and also see to what extent this theory applies to controlled nonholonomic systems with or without symmetry, following [18] and [36], and to degenerate and implicit controlled Lagrangian and Hamiltonian systems, following [6] and [35].

Acknowledgments. The authors thank Hernán Cendra for helpful discussions and the anonymous referees for constructive comments.

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